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NOTE ON A SPECIAL SYMMETRICAL DETERMINANT.

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1. IN the Cambridge and Dublin Mathematical Journal, Vol. I, p. 286 (1846), a correspondent, signing himself H (1), gives the identity

$$\begin{aligned}&(a_1a_2-b_1b_2-c_1c_2)(b_1b_2-c_1c_2-a_1a_2)(c_1c_2-a_1a_2-b_1b_2)\\ &-(a_1a_2-b_1b_2-c_1c_2)(b_1c_2+b_2c_1)^2-(b_1b_2-c_1c_2-a_1a_2)(a_1c_2+a_2c_1)^2\\ &\qquad\qquad-(c_1c_2-a_1a_2-b_1b_2)(a_1b_2+a_2b_1)^2\\ &+2(b_1c_2+b_2c_1)(a_1c_2+a_2c_1)(a_1b_2+a_2b_1)\\ &\qquad=(a_1^2+b_1^2+c_1^2)(a_1a_2+b_1b_2+c_1c_2)(a_2^2+b_2^2+c_2^2).\end{aligned}$$

No proof is added, and at first sight it might appear as if the verification of the identity would be a trifle laborious. The object of the present note is to give a proof interesting to some extent in itself and also as showing how a generalization of the identity may be effected.

2. The left hand member is expressible by a determinant, viz.,

$$\begin{vmatrix} a_1a_2-b_1b_2-c_1c_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 \\ a_1b_2+a_2b_1 & b_1b_2-c_1c_2-a_1a_2 & b_1c_2+b_2c_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & c_1c_2-a_1a_2-b_1b_2 \end{vmatrix},$$

or, if we write S_3 for $a_1a_2+b_1b_2+c_1c_2$, by

$$\begin{vmatrix} 2a_1a_2-S_3 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 \\ a_1b_2+a_2b_1 & 2b_1b_2-S_3 & b_1c_2+b_2c_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2-S_3 \end{vmatrix}.$$

Expanding this according to descending powers of S_3 we have

$$-S_3^3+S_3^2(2a_1a_2+2b_1b_2+2c_1c_2)$$

$$-S_3\left\{\begin{vmatrix} 2b_1b_2 & b_1c_2+b_2c_1 \\ b_1c_2+b_2c_1 & 2c_1c_2 \end{vmatrix}+\begin{vmatrix} 2a_1a_2 & a_1c_2+a_2c_1 \\ a_1c_2+a_2c_1 & 2c_1c_2 \end{vmatrix}\right.$$

$$+\left|\begin{matrix}2a_1a_2 & a_1b_2+a_2b_1\\ a_1b_2+a_2b_1 & 2b_1b_2\end{matrix}\right|\Bigg\}$$

$$+\left|\begin{matrix}2a_1a_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1\\ a_1b_2+a_2b_1 & 2b_1b_2 & b_1c_2+b_2c_1\\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2\end{matrix}\right|$$

where the term independent of S_3

$$=\left|\begin{matrix}a_1 & a_2 & 0\\ b_1 & b_2 & 0\\ c_1 & c_2 & 0\end{matrix}\right|\times\left|\begin{matrix}a_2 & a_1 & 0\\ b_2 & b_1 & 0\\ c_2 & c_1 & 0\end{matrix}\right|=0$$

and the coefficient of the first power of S_3

$$=\left|\begin{matrix}b_1 & c_1\\ b_2 & c_2\end{matrix}\right|^2+\left|\begin{matrix}a_1 & c_1\\ a_2 & c_2\end{matrix}\right|^2+\left|\begin{matrix}a_1 & b_1\\ a_2 & b_2\end{matrix}\right|^2=\left|\begin{matrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{matrix}\right|^2$$

$$=\left|\begin{matrix}a_1^2+b_1^2+c_1^2 & a_1a_2+b_1b_2+c_1c_2\\ a_1a_2+b_1b_2+c_1c_2 & a_2^2+b_2^2+c_2^2\end{matrix}\right|$$

$$=(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-S_3^2.$$

The original determinant is thus found

$$=-S_3^3+S_3^2(2S_3)+S_3[(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-S_3^2]$$

$$=S_3(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2) \tag{1}$$

as was to be proved.

3. A glance at the steps of this proof suffices to suggest a direction in which the theorem may be extended. Writing S_4 for $a_1a_2+b_1b_2+c_1c_2+d_1d_2$ we have

$$\left|\begin{matrix}2a_1a_2-S_4 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & a_1d_2+a_2d_1\\ a_1b_2+a_2b_1 & 2b_1b_2-S_4 & b_1c_2+b_2c_1 & b_1d_2+b_2d_1\\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2-S_4 & c_1d_2+c_2d_1\\ a_1d_2+a_2d_1 & b_1d_2+b_2d_1 & c_1d_2+c_2d_1 & 2d_1d_2-S_4\end{matrix}\right|$$

$$=\left.\begin{matrix}S_4^4-S_4^3(2a_1a_2+2b_1b_2+2c_1c_2+2d_1d_2)\\ +S_4^2(-|a_1b_2|^2-|a_1c_2|^2-|a_1d_2|^2-|b_1c_2|^2-|b_1d_2|^2-|c_1d_2|^2)\\ -S_4(0+0+0+0)+0\end{matrix}\right\}$$

$$=-S_4^4+S_4^2[(a_1a_2+b_1b_2+c_1c_2+d_1d_2)^2-(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2)]$$

$$=-(a_1a_2+b_1b_2+c_1c_2+d_1d_2)^2(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2). \tag{2}$$

The transition from these two cases to the general theorem dealing with two sets of n letters can now be accomplished.

4. Putting $d_1=d_2=0$ in (2) and dividing both members by $-(a_1a_2+b_1b_2+c_1c_2)$ we obtain (1); and similarly any case may be derived from that which follows it.

5. Putting $a_1 = a_2 = a$, $b_1 = b_2 = b$, $c_1 = c_2 = c$, $d_1 = d_2 = d$, in (2) we have

$$\begin{vmatrix} a^2-b^2-c^2-d^2 & 2ab & 2ac & 2ad \\ 2ab & b^2-c^2-d^2-a^2 & 2bc & 2bd \\ 2ac & 2bc & c^2-d^2-a^2-b^2 & 2cd \\ 2ad & 2bd & 2cd & d^2-a^2-b^2-c^2 \end{vmatrix} = -(a^2+b^2+c^2+d^2)^4$$

and from this, by making $d = 0$, there results

$$\begin{vmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-c^2-a^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{vmatrix} = (a^2+b^2+c^2)^3$$

and thence in the same way

$$\begin{vmatrix} a^2+b^2 & 2ab \\ 2ab & b^2-a^2 \end{vmatrix} = -(a^2+b^2)^2,$$

the identity well known in connection with Euc. I. 47, giving the sum of two squares as a square.

THE BITANGENTIAL.

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THE curve which passes through the points of contact of bitangents of a given curve is called the *bitangential* of that curve.

Such a curve may be determined by the method of problem 331, ANALYST. It is, however, desirable to obtain a curve of lower order, and for this purpose Salmon has given two methods in Higher Plane Curves.

Let us put

$$A = \begin{vmatrix} \frac{d^2u}{dy^2} & \frac{d^2u}{dz\,dy} \\ \frac{d^2u}{dy\,dz} & \frac{d^2u}{dz^2} \end{vmatrix}, \quad B = \begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2u}{dz\,dx} \\ \frac{d^2u}{dx\,dz} & \frac{d^2u}{dz^2} \end{vmatrix}, \quad C = \begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2u}{dydx} \\ \frac{d^2u}{dxdy} & \frac{d^2u}{dy^2} \end{vmatrix},$$

$$D = \begin{vmatrix} \frac{d^2u}{dx\,dz} & \frac{d^2u}{dz\,dy} \\ \frac{d^2u}{dx^2} & \frac{d^2u}{dxdy} \end{vmatrix}, \quad E = \begin{vmatrix} \frac{d^2u}{dxdy} & \frac{d^2u}{dx\,dz} \\ \frac{d^2u}{dy^2} & \frac{d^2u}{dy\,dz} \end{vmatrix}, \quad F = \begin{vmatrix} \frac{d^2u}{dy\,dz} & \frac{d^2u}{dxdy} \\ \frac{d^2u}{dz^2} & \frac{d^2u}{dx\,dz} \end{vmatrix};$$